Internet Appendix for
"Treasury Bill Shortages and the Pricing of Short-Term Assets"

ADRIEN D'AVERNAS and QUENTIN VANDEWEYER*

We describe the liquidity management problem of banks as a discrete-time problem with an interim period in which assets can only be traded at some cost. We then show that this problem converges to the continuous-time problems of traditional and shadow banks from Section I of the main article. This micro-foundation draws inspiration from Bianchi and Bigio (2022) (adding fire sales and liquid money market asset holdings) and He and Xiong (2012) (adding reserves and liquid money market asset holdings). This micro-foundation is also similar to that of d'Avernas, Vandeweyer, and Darracq-Pariès (2019) when adding T-bills and repo transactions and not allowing for interbank trade during the illiquid stage.

Timing Time is discrete with an infinite horizon. Each period is divided into two stages: the liquid stage ℓ and the illiquid stage i. Both stages last a period of time Δt. In the liquid stage, there is no liquidity friction and portfolios can be adjusted at market prices without any cost. Then, the macroeconomic shock on risky securities realizes and interest rates are paid. At the beginning of the illiquid stage, deposits are randomly reshuffled from some banks—*the deficit banks*—to others—*the surplus banks*. Deficit banks cannot contract new loans and have to rely on disbursing existing assets in order to settle their debts with the surplus banks. There are two types of liquidity frictions in the illiquid stage. First, only a fraction of assets can be mobilized to settle debts. Second, it is costly to use assets during the illiquid stage for settlement purposes. This cost depends on the liquidity of the assets, with risky securities being the most illiquid. After the end of the illiquid stage, the economy enters a new liquid stage for the next period.

The Liquid Stage In the liquid stage, all banks can trade assets without friction. The law of motion for the wealth of banks in the liquid stage can then be written as

$$\Delta^\ell n_t = \Big(r_t^m m_t + r_t^i i_t + r_t^b b_t + r_t^p p_t - r_t^d d_t - c_t n_t + \mu_t^\tau n_t\Big)\Delta t. \tag{IA.1}$$

Bankers face a portfolio choice problem with four different assets: securities portfolio s_t, Treasury bills b_t, central bank reserves m_t, interbank lending i_t, and deposits d_t. In equation (IA.1), r_t^i is the return on an illiquid asset, r_t^m the interest rate paid by the central bank on

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its reserves, r^b_t the interest rate paid by the government on T-bills, r^p_t the interest rate on repos, and r^d_t the interest rate on deposits. Banks also choose their consumption rate c_t as a fraction of their wealth and receive a flow of transfers per unit of wealth of μ^τ_t.

The Illiquid Stage Each individual bank is subject to an *idiosyncratic* deposit shock,

$$\Delta^i d_t = \sigma^d_t d_t \varepsilon^i_t \sqrt{\Delta t},$$

where ε^i_t is a binomial stochastic variable distributed with even probabilities:

$$\varepsilon^i_t = \begin{cases} +1 & \text{with } p = 1/2, \\ -1 & \text{with } p = 1/2. \end{cases}$$

In the illiquid period, interbank loans i_t cannot be contracted. The balance sheet constraint of the bank imposes that the flow of deposits is matched with an equivalent flow of securities. That is,

$$\Delta^i m_t + \Delta^i p_t + \Delta^i i_t + \Delta^i b_t = \Delta^i d_t.$$

The flows of assets $\Delta^i s_t$, $\Delta^i i_t$, $\Delta^i p_t$, $\Delta^i m_t$, and $\Delta^i b_t$ are chosen by deficit banks to minimize the net cost of transactions. To simplify the model, we assume that the costs of trading illiquid assets are fixed exogenously[1] and transferred from deficit to surplus banks. We capture these costs with parameters λ^s, λ^m, λ^p, λ^i, and λ^b. Surplus banks do not face liquidity constraints and take these opportunities to purchase these assets at a discounted price as given. Because the policy functions are linear in agents' wealth, the distribution of these flows does not impact the recursive competitive equilibrium.

We can then write the net impact of the cost of the deposit shock on an individual bank's wealth as

$$\Delta^i n_t = \lambda^i \Delta^i i_t + \lambda^m \Delta^i m_t + \lambda^p \Delta^i p_t + \lambda^b \Delta^i b_t.$$

[1]We do not provide a micro-foundation for the cost of a fire sale, but we refer to the large literature in which it arises either as a consequence of the shift in bargaining power under strong selling pressure (see Brunnermeier and Pedersen (2005), Duffie, Gârleanu, and Pedersen (2005), Duffie, Gârleanu, and Pedersen (2007), Duffie and Strulovici (2012)) or asymmetry of information (see Wang (1993), Malherbe (2014)). The intuition is that using reserves or other liquid money market assets has a negligible cost compared with having to sell risky securities. The intuition for including short-maturity loans as liquid assets is that if the illiquid stage lasts for a longer period than the maturity of the short-term loan, the bank will be able to use the funds lent at the due date, thereby creating a liquidity component of the term structure as modeled by Acharya and Skeie (2011) and documented empirically by Greenwood, Hanson, and Stein (2015).

Substituting for the balance sheet constraint, we have

$$\Delta^i n_t = \lambda^m \Delta^i m_t + \lambda^p \Delta^i p_t + \lambda^b \Delta^i b_t + \lambda^i \left(\Delta^i d_t - \Delta^i m_t - \Delta^i p_t - \Delta^i b_t \right),$$

which can be rewritten as

$$\Delta^i n_t = \lambda^i \left(\Delta^i d_t - \frac{\lambda^i - \lambda^m}{\lambda^i} \Delta^i m_t - \frac{\lambda^i - \lambda^p}{\lambda^i} \Delta^i p_t - \frac{\lambda^i - \lambda^b}{\lambda^i} \Delta^i b_t \right). \tag{IA.2}$$

Moreover, a second type of liquidity friction constrains the number of assets that can be sold by deficit banks during the time interval Δt. A deficit bank can only decrease its asset holdings and only up to a certain threshold. In order to converge to a Brownian motion in the continuous time approximation, this amount is proportional to $\sqrt{\Delta t}$. For example, a deficit bank cannot sell more than a fraction $\delta^s \sqrt{\Delta t}$ of its risky securities over the interval Δt. We write these constraints as

$$0 \geq \Delta^i i_t \geq -\delta^i i_t \sqrt{\Delta t}, \tag{IA.3}$$

$$0 \geq \Delta^i m_t \geq -\delta^m m_t \sqrt{\Delta t}, \tag{IA.4}$$

$$0 \geq \Delta^i p_t \geq -\delta^p p_t \sqrt{\Delta t}, \tag{IA.5}$$

$$0 \geq \Delta^i b_t \geq -\delta^b b_t \sqrt{\Delta t}. \tag{IA.6}$$

The optimization problem of deficit banks in the illiquid stage amounts to the static[2] minimization of their losses under the liquidity constraints

$$\min_{\Delta^i p_t, \Delta^i m_t, \Delta^i i_t, \Delta^i b_t} \Delta^i n_t,$$

where $\Delta^i n_t$ is given by (IA.2), $\Delta^i d_t = -\sigma_t^d \sqrt{\Delta t}$ and is subject to previously stated liquidity frictions.

We first consider the case in which liquid assets are not sufficient for a deficit bank to cover its funding needs, that is, $\sigma_t^d d_t > \delta^m m_t + \delta^p p_t + \delta^b b_t$. Since using illiquid assets i_t is the most costly asset, deficit banks always first use their liquid assets m_t, b_t, and p_t and only then resort to selling securities in order to settle remaining due debt positions. Hence, the

[2]The problem is static since banks are able to fully readjust their balance sheets at the beginning of the next period.

optimal portfolio adjustments are given by

$$\Delta^i i_t = \Delta^i d_t + \Delta^i m_t + \Delta^i p_t + \Delta^i b_t,$$
$$\Delta^i m_t = -\delta^m m_t \sqrt{\Delta t},$$
$$\Delta^i p_t = -\delta^i p_t \sqrt{\Delta t},$$
$$\Delta^i b_t = -\delta^b b_t \sqrt{\Delta t}.$$

Intuitively, to avoid having to fire-sale illiquid securities at a cost λ^s, deficit banks mobilize as much as they can from their other (more liquid) asset holdings. Note that all losses from a deficit bank are gained by a surplus bank. Therefore, assuming that $\sigma_t^d d_t > \delta^m m_t + \delta^p p_t + \delta^b b_t$, the law of motion of bank wealth in the illiquid stage can be written as

$$\Delta^i n_t = \lambda^i \Big(\sigma_t^d d_t - \theta^m m_t - \theta^p p_t - \theta^b b_t \Big) \varepsilon_t^i \sqrt{\Delta t},$$

where $\theta^j \equiv \frac{\lambda^j - \lambda^s}{\lambda^s} \delta^j$ for $j \in \{m, p, b\}$ is defined as the liquidity index of a given asset, taking into account the liquidity frictions on prices and on quantities.

Let's now consider the case in which liquidity is sufficient to cover a negative funding shock: $\sigma_t^d d_t \leq \delta^m m_t + \delta^p p_t + \delta^b b_t$. In this case, the deficit bank does not have to pay any securities' fire-sale cost but still has to cover the cost of using liquid assets. Computing this cost requires knowing which assets have been used. Using a similar logic as previously, the deficit bank will always first use less costly assets. To avoid dealing with multiple kinks and keep the model tractable in its continuous-time approximation, we make the following technical assumption.

ASSUMPTION IA.1 (Costless Liquidity Absent Fire-sale Risk): *When there is no fire-sale risk, $\sigma_t^d d_t \leq \delta^m m_t + \delta^p p_t + \delta^b b_t$, there is no cost of mobilizing liquid assets $\lambda^m = \lambda^b = \lambda^p = 0$.*

When Assumption IA.1 holds, the threshold at which banks do not have to fire sale securities corresponds to the threshold at which liquidity risk is nil and the law of motion for the wealth of banks is given by

$$\Delta^i n_t = 0.$$

Continuous-Time Approximation We can combine the law of motion of both stages to get

$$
\begin{aligned}
\Delta n_t &= \Delta^{\ell} n_t + \Delta^{i} n_t \\
&= \Big(r_t^i i_t + r_t^m m_t + r_t^p p_t + r_t^b b_t - r_t^d d_t - c_t n_t + \mu_t^\tau n_t \Big) \Delta t \\
&+ \lambda^s \max \left\{ \sigma_t^d d_t - \theta^m m_t - \theta^p p_t - \theta^b b_t, 0 \right\} \varepsilon_t^i \sqrt{\Delta t}.
\end{aligned}
$$

Finally, the limit when Δt tends to zero is given by

$$
\begin{aligned}
dn_t &= \Big(r_t^i i_t + r_t^m m_t + r_t^p p_t + r_t^b b_t - r_t^d d_t - c_t n_t + \mu_t^\tau n_t \Big) dt \\
&+ \lambda^s \max \left\{ \sigma_t^d d_t - \theta^m m_t - \theta^p p_t - \theta^b b_t, 0 \right\} d\widetilde{Z}_t,
\end{aligned}
$$

where $\widetilde{Z}_t$ is an idiosyncratic Brownian motion.

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